

# ON THE HILBERT FUNCTION ON $\mathbb{P}^1 \times \mathbb{P}^1$

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**ABSTRACT.** Let  $Q = \mathbb{P}^1 \times \mathbb{P}^1$  and let  $X \subset Q$  be a 0-dimensional scheme. This paper is a first step towards the characterization of Hilbert functions of 0-dimensional schemes in  $Q$ . In particular we show how, under some conditions on  $X$ , its Hilbert function changes when we add points to  $X$  lying on a  $(1, 0)$  or  $(0, 1)$ -line. As a particular case we show also that if  $X$  is ACM this result holds without any additional hypothesis.

## 1. INTRODUCTION

Let  $Q = \mathbb{P}^1 \times \mathbb{P}^1$  and let  $X \subset Q$  be a 0-dimensional scheme. Let  $R$  and  $C$  be, respectively, a  $(1, 0)$  and a  $(0, 1)$ -line not containing any point of  $X$  and let  $Z$  be a 0-dimensional scheme given by  $X$  and some points on  $R$  or  $C$ . In this paper we deal with the problem of finding the Hilbert matrix (function) of  $Z$  with respect to the Hilbert matrix of  $X$ . A first approach was given in a very particular case in 1992 in [2], with the only perspective of comprehending the Hilbert functions of ACM 0-dimensional schemes in  $Q$ . This paper is the first real step towards the characterization of the Hilbert functions of 0-dimensional schemes in  $Q$  that are not ACM.

In Theorem 3.1 and Theorem 3.2 we improve the result in [2], under some geometric and algebraic conditions that, as we see in Example 3.5, can not be suppressed without any further assumption. In Theorem 4.2 and Theorem 4.3 we see that the result holds for all ACM schemes  $X$  without any additional condition. As an application in Section 5 we compute the Hilbert matrix of any non ACM reduced set of points in  $Q$  having a certain position in a grid of  $(1, 0)$  and  $(0, 1)$ -lines. This previously could be done and was known just for ACM 0-dimensional schemes.

A good reference for a general discussion on 0-dimensional schemes on  $\mathbb{P}^1 \times \mathbb{P}^1$  is [2], in which there are the most important results about the Hilbert function. Further results on the Hilbert function has been obtained just in the particular case of fat points (see for example [4] and [6]).

## 2. NOTATION AND PRELIMINARY RESULTS

Let  $k$  be an algebraically closed field, let  $\mathbb{P}^1 = \mathbb{P}_k^1$ , let  $Q = \mathbb{P}^1 \times \mathbb{P}^1$  and let  $\mathcal{O}_Q$  be its structure sheaf. For any sheaf  $\mathcal{F}$  we denote:

$$\mathcal{F}(a, b) = \mathcal{F} \otimes \mathcal{O}_Q(a, b).$$

Let us consider the bi-graded ring:

$$S = H_*(\mathcal{O}_Q) = \bigoplus_{a, b \geq 0} H^0(\mathcal{O}_Q(a, b)).$$

For any bi-graded  $S$ -module  $N$  let  $N_{(i, j)}$  the component of degree  $(i, j)$ .

Given  $X \subset Q$  0-dimensional scheme, let  $I(X) \subset S$  be the associated saturated sheaf,  $S(X) = S/I(X)$  and  $\mathcal{I}_X \subset \mathcal{O}_Q$  its ideal sheaf.

**Definition 2.1.** The function:

$$M_X: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{N}$$

defined by:

$$M_X(i, j) = \dim_k S(X)_{(i,j)} = \dim_k S_{(i,j)} - \dim_k I(X)_{(i,j)}$$

is called *Hilbert function* of  $X$ . The function  $M_X$  can be represented as a matrix with infinite integers entries:

$$M_X = (M_X(i, j)) = (m_{ij})$$

called *Hilbert matrix* of  $X$ .

Note that  $M_X(i, j) = 0$  for either  $i < 0$  or  $j < 0$  and so we restrict ourselves to the range  $i \geq 0$  and  $j \geq 0$ . Moreover, for  $i \gg 0$  and  $j \gg 0$   $M_X(i, j) = \deg X$ .

**Definition 2.2.** Given the Hilbert matrix  $M_X$  of a 0-dimensional scheme  $X \subset Q$ , the *first difference of the Hilbert function* of  $X$  is the matrix:

$$\Delta M_X = (c_{ij}),$$

where:

$$c_{ij} = m_{ij} - m_{i-1j} - m_{ij-1} + m_{i-1j-1}.$$

We consider the following matrices:

$$\Delta^R M_X = (a_{ij}) \text{ and } \Delta^C M_X = (b_{ij}),$$

with  $a_{ij} = m_{ij} - m_{ij-1}$  and  $b_{ij} = m_{ij} - m_{i-1j}$ . Note that:

$$c_{ij} = a_{ij} - a_{i-1j} = b_{ij} - b_{ij-1}$$

and

$$m_{ij} = \sum_{\substack{h \leq i \\ k \leq j}} c_{hk}.$$

**Theorem 2.3** ([2, Theorem 2.11]). *Given a 0-dimensional scheme  $X \subset Q$  and given its Hilbert matrix  $M_X$ , the first difference  $\Delta M_X = (c_{ij})$  satisfies the following conditions:*

- (1)  $c_{ij} \leq 1$  and  $c_{ij} = 0$  for  $i \gg 0$  or  $j \gg 0$ ;
- (2) if  $c_{ij} \leq 0$ , then  $c_{rs} \leq 0$  for any  $(r, s) \geq (i, j)$ ;
- (3) for every  $(i, j)$   $0 \leq \sum_{t=0}^j c_{it} \leq \sum_{t=0}^j c_{i-1t}$  and  $0 \leq \sum_{t=0}^i c_{tj} \leq \sum_{t=0}^i c_{tj-1}$ .

**Remark 2.4.** If  $X \subset Q$  is a 0-dimensional scheme, let us consider  $a = \min\{i \in \mathbb{N} \mid I(X)_{(i,0)} \neq 0\} - 1$  and  $b = \min\{j \in \mathbb{N} \mid I(X)_{(0,j)} \neq 0\} - 1$ . Then by Theorem 2.3  $\Delta M_X$  is zero out of the rectangle with opposite vertices  $(0, 0)$  and  $(a, b)$ . In this case we say that  $\Delta M_X$  is of syze  $(a, b)$ .

Let  $M_X = (m_{ij})$  be the Hilbert matrix of a 0-dimensional scheme  $X \subset Q$ . Using the notation in [2], for every  $j \geq 0$  we set:

$$i(j) = \min\{t \in \mathbb{N} \mid m_{tj} = m_{t+1j}\} = \min\{t \in \mathbb{N} \mid b_{t+1j} = 0\}.$$

and for every  $i \geq 0$  we set:

$$j(i) = \min\{t \in \mathbb{N} \mid m_{it} = m_{it+1}\} = \min\{t \in \mathbb{N} \mid a_{it+1} = 0\}$$

In particular, we see that  $i(0) = a$  and  $j(0) = b$ .

Let  $X \subset Q$  be a 0-dimensional scheme and let  $L$  be a line defined by a form  $l$ . Let  $J = (I(X), l)$  and let  $d = \deg(\text{sat } J)$ . Then we call  $d$  the number of points of  $X$  on the line  $L$  and, by abuse of notation, we make the position  $d = \#(X \cap L)$ . We say that  $L$  is disjoint from  $X$  if  $d = 0$ .

The key result used in Section 3 is the following:

**Theorem 2.5** ([2, Theorem 2.12]). *Let  $X \subset Q$  be a 0-dimensional scheme and let  $M_X = (m_{ij})$  be its Hilbert matrix. Then for every  $j \geq 0$  there are just  $a_{i(0)j} - a_{i(0)j+1}$  lines of type  $(1, 0)$  each containing just  $j+1$  points of  $X$  and, similarly, for every  $i \geq 0$  there are just  $b_{ij(0)} - b_{i+1j(0)}$  lines of type  $(0, 1)$  each containing just  $i+1$  points of  $X$ .*

The result that in this paper we improve is given by the following:

**Theorem 2.6** ([2, Lemma 2.15]). *Let  $X \subset Q$  be a 0-dimensional scheme and let  $M_X$  be its Hilbert matrix. Let  $R_0, \dots, R_a$  and  $C_0, \dots, C_b$  be, respectively, the  $(1, 0)$  and  $(0, 1)$ -lines containing  $X$  and at least one point of  $X$ . Let  $R$  be a  $(1, 0)$ -line disjoint from  $X$  and let  $Y = X \cup R \cap (C_0 \cup \dots \cup C_n)$ , with  $n \geq b$  and  $C_{b+1}, \dots, C_n$  arbitrary  $(0, 1)$ -lines. Then:*

$$\Delta M_Y^{(i,j)} = \begin{cases} 1 & \text{for } i = 0, j \leq n \\ 0 & \text{for } i = 0, j \geq n+1 \\ \Delta M_X^{(i-1,j)} & \text{for } i \geq 1. \end{cases}$$

Of course a similar result can be proved by adding  $m+1$  points on a  $(0, 1)$ -line  $C$  disjoint from  $X$ . So, with the previous notation, it is possible to prove the following result.

**Theorem 2.7.** *Let  $C$  be a  $(0, 1)$ -line disjoint from  $X$ . Let  $Y = X \cup C \cap (R_0 \cup \dots \cup R_m)$ ,  $m \geq a$ , and  $R_{a+1}, \dots, R_m$  arbitrary  $(1, 0)$ -lines. Then:*

$$\Delta M_Y^{(i,j)} = \begin{cases} 1 & \text{for } i \leq m, j = 0 \\ 0 & \text{for } i \geq m+1, j = 0 \\ \Delta M_X^{(i,j-1)} & \text{for } j \geq 1. \end{cases}$$

### 3. THE FIRST DIFFERENCE OF THE HILBERT FUNCTION

Let  $X \subset Q$  be a 0-dimensional scheme and let  $M_X$  be its Hilbert matrix. In all this paper we suppose that  $\Delta M_X$  is of size  $(a, b)$  and we denote by  $R_0, \dots, R_a$  and  $C_0, \dots, C_b$ , respectively, the  $(1, 0)$  and  $(0, 1)$ -lines containing  $X$  and at least one point of  $X$ .

**Theorem 3.1.** *Let  $R$  be a  $(1, 0)$ -line disjoint from  $X$ . Let  $C_{b+1}, \dots, C_n$ ,  $n \geq b$ , be arbitrary  $(0, 1)$ -lines and  $i_1, \dots, i_r \in \{0, \dots, b\}$ . Let  $\mathcal{P} = \{R \cap C_i \mid i \in \{0, \dots, n\}, i \neq i_1, \dots, i_r\}$  and let  $Z = X \cup \mathcal{P}$ . Suppose also that on the  $(0, 1)$ -line  $C_{i_k}$  there are  $q_k$  points of  $X$  for  $k = 1, \dots, r$  and that  $q_1 \leq q_2 \leq \dots \leq q_r$ . Then, given  $T = \{(q_1, n), (q_2, n-1), \dots, (q_r, n-r+1)\}$ , we have:*

$$\Delta M_Z^{(i,j)} = \begin{cases} 1 & \text{if } i = 0, j \leq n \\ 0 & \text{if } i = 0, j \geq n+1 \\ \Delta M_X^{(i-1,j)} & \text{if } i \geq 1 \text{ and } (i, j) \notin T \\ \Delta M_X^{(i-1,j)} - 1 & \text{if } i \geq 1 \text{ and } (i, j) \in T \end{cases}$$

if one of the following conditions holds:

- (1)  $r = 1$ ;
- (2)  $r \geq 2$ ,  $q_{r-1} < q_r$  and for any  $k \in \{1, \dots, r-1\}$  and  $i \geq q_k$   $\Delta M_X^{(i, n-k+1)} = 0$ ;
- (3)  $r \geq 2$ ,  $q_{r-1} = q_r$  and for any  $k \in \{1, \dots, r\}$  and  $i \geq q_k$   $\Delta M_X^{(i, n-k+1)} = 0$ .

*Proof.* Let  $Y = X \cup (R \cap (\bigcup_{i=0}^n C_i))$ . By Theorem 2.6 it is sufficient to prove that:

$$\Delta M_Z(i, j) = \begin{cases} \Delta M_Y(i, j) & \text{if } (i, j) \notin T \\ \Delta M_Y(i, j) - 1 & \text{if } (i, j) \in T. \end{cases}$$

We divide the proof in different steps.

**Step 1.**  $\Delta M_Z^{(0, j)} = \Delta M_Y^{(0, j)} = 1$  for  $j \leq n$ ,  $\Delta M_Z^{(0, j)} = \Delta M_Y^{(0, j)} = 0$  for  $j \geq n+1$  and  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)} = \Delta M_X^{(i-1, j)}$  for any  $(i, j)$  with  $j < n-r+1$  and  $i \geq 1$ .

It is easy to see that  $\Delta M_Z^{(0, j)} = 1$  for  $j \leq n$ , because for such values of  $j$   $h^0(\mathcal{J}_Z(0, j)) = 0$ . Moreover,  $\Delta M_Z^{(0, j)} = 0$  for  $j \geq n+1$  by Remark 2.4.

Taken  $(i, j)$ , with  $j < n-r+1$  and  $i \geq 1$ , any  $(i, j)$ -curve containing  $Z$  must contain  $R$  and so  $h^0(\mathcal{J}_Z(i, j)) = h^0(\mathcal{J}_X(i-1, j))$  and  $\Delta M_Z^{(i, j)} = \Delta M_X^{(i-1, j)}$ .

Let  $r_1, \dots, r_{t+1}$  be a sequence of positive integers such that  $q_1 = \dots = q_{r_1} < q_{r_1+1} = \dots = q_{r_2} < \dots < q_{r_t+1} = \dots = q_{r_{t+1}} = q_r$  and let  $r_0 = 0$ .

**Step 2.** If  $h \in \{1, \dots, t+1\}$ , then  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)}$  for any  $(i, j) \leq (q_{r_{h-1}+1} - 1, n - r_{h-1})$  and for  $(i, j) = (q_{r_{h-1}+1}, j)$  with  $j < n - r_h + 1$ .

Taken  $(i, j) \leq (q_{r_{h-1}+1} - 1, n - r_{h-1})$ , then any  $(i, j)$ -curve containing  $Z$  must contain  $C_{i_{r_{h-1}+1}}, \dots, C_{i_r}$  and so it must contain  $R$ . This means that  $h^0(\mathcal{J}_Z(i, j)) = h^0(\mathcal{J}_Y(i, j))$  and so that  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)}$ .

Taken  $(i, j) = (q_{r_{h-1}+1}, j)$  with  $j < n - r_h + 1$ , then any  $(i, j)$ -curve containing  $Z$  must contain  $C_{i_{r_{h-1}+1}}, \dots, C_{i_r}$  and so it must contain  $R$ . Again this implies  $h^0(\mathcal{J}_Z(i, j)) = h^0(\mathcal{J}_Y(i, j))$  and  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)}$ .

**Step 3.** For any  $1 \leq h \leq t+1$  one of the following conditions holds:

- (1) there exists  $\bar{j}$  with  $n - r_h + 1 \leq \bar{j} \leq n - r_{h-1}$  such that  $\Delta M_Z^{(q_{r_{h-1}+1}, \bar{j})} = \Delta M_Y^{(q_{r_{h-1}+1}, \bar{j})}$  for any  $j < \bar{j}$  and  $\Delta M_Z^{(q_{r_{h-1}+1}, \bar{j})} < \Delta M_Y^{(q_{r_{h-1}+1}, \bar{j})}$ ;
- (2)  $\Delta M_Z^{(q_{r_{h-1}+1}, j)} = \Delta M_Y^{(q_{r_{h-1}+1}, j)}$  for any  $n - r_h + 1 \leq j \leq n - r_{h-1}$ .

Since  $Z \subset Y$  we see that  $M_Z(q_{r_{h-1}+1}, n - r_h + 1) \leq M_Y(q_{r_{h-1}+1}, n - r_h + 1)$ . Moreover, by Step 2 we see that  $M_Z(i, j) = M_Y(i, j)$  for any  $(i, j) < (q_{r_{h-1}+1}, n - r_h + 1)$ . This implies that:

$$\Delta M_Z^{(q_{r_{h-1}+1}, n-r_h+1)} \leq \Delta M_Y^{(q_{r_{h-1}+1}, n-r_h+1)}.$$

If  $\Delta M_Z^{(q_{r_{h-1}+1}, n-r_h+1)} = \Delta M_Y^{(q_{r_{h-1}+1}, n-r_h+1)}$ , then we can repeat the previous procedure to show that  $\Delta M_Z^{(q_{r_{h-1}+1}, n-r_h+2)} \leq \Delta M_Y^{(q_{r_{h-1}+1}, n-r_h+2)}$ . By iterating this procedure we get the conclusion of Step 3.

**Step 4.**

- (1) If  $r \geq 2$  and  $q_{r-1} = q_r$ , given  $h \in \{1, \dots, t+1\}$  and  $j \in \{n - r_h + 1, \dots, n - r_{h-1}\}$ , we have:

$$\sum_{i=q_{r_{h-1}}+1}^{a+1} \Delta M_Z^{(i,j)} = \Delta M_Y^{(q_{r_{h-1}}+1,j)} - 1;$$

- (2) If  $r \geq 2$  and  $q_{r-1} < q_r$ , given  $h \in \{1, \dots, t\}$  and  $j \in \{n - r_h + 1, \dots, n - r_{h-1}\}$ , we have:

$$\sum_{i=q_{r_{h-1}}+1}^{a+1} \Delta M_Z^{(i,j)} = \Delta M_Y^{(q_{r_{h-1}}+1,j)} - 1$$

and

$$\sum_{i=q_r}^{a+1} \Delta M_Z^{(i,n-r+1)} = \sum_{i=q_r}^{a+1} \Delta M_Y^{(i,n-r+1)} - 1.$$

Let us first note that by Theorem 2.5:

$$a_{i(0)n-r}(Z) - a_{i(0)n-r+1}(Z) = \sum_{i \leq a+1} \Delta M_Z^{(i,n-r)} - \sum_{i \leq a+1} \Delta M_Z^{(i,n-r+1)}$$

is equal to the number of  $(1,0)$ -lines containing precisely  $n - r + 1$  points of  $Z$ , while:

$$a_{i(0)n-r}(Y) - a_{i(0)n-r+1}(Y) = \sum_{i \leq a+1} \Delta M_Y^{(i,n-r)} - \sum_{i \leq a+1} \Delta M_Y^{(i,n-r+1)}$$

is equal to the number of  $(1,0)$ -lines containing precisely  $n - r + 1$  points of  $Y$ . By hypothesis it must be:

$$\begin{aligned} \sum_{i \leq a+1} \Delta M_Z^{(i,n-r)} - \sum_{i \leq a+1} \Delta M_Z^{(i,n-r+1)} &= \\ &= \sum_{i \leq a+1} \Delta M_Y^{(i,n-r)} - \sum_{i \leq a+1} \Delta M_Y^{(i,n-r+1)} + 1. \end{aligned}$$

By Step 1 this implies that:

$$\sum_{i \leq a+1} \Delta M_Z^{(i,n-r+1)} = \sum_{i \leq a+1} \Delta M_Y^{(i,n-r+1)} - 1.$$

Let us now suppose that for some  $j \geq n - r + 1$ , with  $j < n$ , we have:

$$(1) \quad \sum_{i \leq a+1} \Delta M_Z^{(i,j)} = \sum_{i \leq a+1} \Delta M_Y^{(i,j)} - 1.$$

We will show that:

$$(2) \quad \sum_{i \leq a+1} \Delta M_Z^{(i,j+1)} = \sum_{i \leq a+1} \Delta M_Y^{(i,j+1)} - 1.$$

Again, by Theorem 2.5  $\sum_{i \leq a+1} \Delta M_Z^{(i,j)} - \sum_{i \leq a+1} \Delta M_Z^{(i,j+1)}$  is equal to the number of  $(1,0)$ -lines containing precisely  $j + 1$  points of  $Z$ , while  $\sum_{i \leq a+1} \Delta M_Y^{(i,j)} - \sum_{i \leq a+1} \Delta M_Y^{(i,j+1)}$  is equal to the number of  $(1,0)$ -lines containing precisely  $j + 1$  points of  $Y$ . By hypothesis it must be:

$$\sum_{i \leq a+1} \Delta M_Z^{(i,j)} - \sum_{i \leq a+1} \Delta M_Z^{(i,j+1)} = \sum_{i \leq a+1} \Delta M_Y^{(i,j)} - \sum_{i \leq a+1} \Delta M_Y^{(i,j+1)}.$$

By (1) it means that (2) holds, so that:

$$\sum_{i \leq a+1} \Delta M_Z^{(i,j)} = \sum_{i \leq a+1} \Delta M_Y^{(i,j)} - 1$$

for any  $j$  with  $n - r + 1 \leq j \leq n$ . Now the hypotheses on  $X$ , Step 1 and Step 2 give us the conclusion of Step 4.

**Step 5.** *If  $r \geq 2$ ,  $\Delta M_Z^{(i,j)} = \Delta M_Y^{(i,j)} = 0$  for any  $(i,j) \geq (q_1 + 1, n - r_1 + 1)$  and  $\Delta M_Z^{(i,j)} = \Delta M_Y^{(i,j)} - 1$  for any  $(i,j) \in \{(q_1, n), (q_1, n - 1), \dots, (q_1, n - r_1 + 1)\}$ .*

By Theorem 2.5 we know that:

$$b_{q_1-1j(0)}(Z) - b_{q_1j(0)}(Z) = \sum_{j \leq n} \Delta M_Z^{(q_1-1,j)} - \sum_{j \leq n} \Delta M_Z^{(q_1,j)}$$

is equal to the number of  $(0, 1)$ -lines containing exactly  $q_1$  points of  $Z$  and, in the same way that:

$$b_{q_1-1j(0)}(Y) - b_{q_1j(0)}(Y) = \sum_{j \leq n} \Delta M_Y^{(q_1-1,j)} - \sum_{j \leq n} \Delta M_Y^{(q_1,j)}$$

is equal to the number of  $(0, 1)$ -lines containing exactly  $q_1$  points of  $Y$ . So by construction we have:

$$(3) \quad \sum_{j \leq n} \Delta M_Z^{(q_1-1,j)} - \sum_{j \leq n} \Delta M_Z^{(q_1,j)} = \sum_{j \leq n} \Delta M_Y^{(q_1-1,j)} - \sum_{j \leq n} \Delta M_Y^{(q_1,j)} + r_1.$$

By what we proved in Step 2 we see that:

$$\sum_{j \leq n} \Delta M_Z^{(q_1-1,j)} = \sum_{j \leq n} \Delta M_Y^{(q_1-1,j)}$$

so that by (3) and again by Step 2 we get:

$$(4) \quad \sum_{n-r_1+1}^n \Delta M_Z^{(q_1,j)} = \sum_{n-r_1+1}^n \Delta M_Y^{(q_1,j)} - r_1.$$

By this equality and by Step 3 we see that there exists  $n - r_1 + 1 \leq \bar{j} \leq n$  such that  $\Delta M_Z^{(q_1,j)} = \Delta M_Y^{(q_1,j)}$  for any  $j < \bar{j}$  and  $\Delta M_Z^{(q_1,\bar{j})} < \Delta M_Y^{(q_1,\bar{j})}$ . In particular,  $\Delta M_Z^{(q_1,\bar{j})} \leq 0$  and so by Theorem 2.3 we have:

$$\Delta M_Z^{(i,\bar{j})} \leq 0$$

for any  $i \geq q_1$ . By Step 4 we have:

$$\begin{aligned} \sum_{i=q_1}^{a+1} \Delta M_Z^{(i,\bar{j})} &= \Delta M_Y^{(q_1,\bar{j})} - 1 \\ \Rightarrow 0 &\geq \sum_{i=q_1+1}^{a+1} \Delta M_Z^{(i,\bar{j})} = \Delta M_Y^{(q_1,\bar{j})} - \Delta M_Z^{(q_1,\bar{j})} - 1 \geq 0. \end{aligned}$$

This means that  $\Delta M_Y^{(q_1,\bar{j})} - \Delta M_Z^{(q_1,\bar{j})} - 1 = 0$  and that  $\Delta M_Z^{(i,\bar{j})} = 0$  for any  $i \geq q_1 + 1$ .

Now take any  $j > \bar{j}$ . By the fact that  $\Delta M_Z^{(i, \bar{j})} = 0$  for any  $i \geq q_1 + 1$  and by Theorem 2.3 we can say that  $\Delta M_Z^{(i, j)} \leq 0$  for any  $i \geq q_1 + 1$  and any  $j > \bar{j}$ . By Step 4 we get:

$$(5) \quad 0 \geq \sum_{i=q_1+1}^{a+1} \Delta M_Z^{(i, j)} = \Delta M_Y^{(q_1, j)} - \Delta M_Z^{(q_1, j)} - 1$$

so that:

$$\Delta M_Z^{(q_1, j)} \geq \Delta M_Y^{(q_1, j)} - 1$$

for any  $j > \bar{j}$ . So we can say that:

$$(6) \quad \sum_{j=n-r_1+1}^n \Delta M_Z^{(q_1, j)} \geq \sum_{j=n-r_1+1}^n \Delta M_Y^{(q_1, j)} - n + \bar{j} - 1.$$

This fact compared to (4) gives us that  $\bar{j} \leq n - r_1 + 1$ , but by hypothesis  $\bar{j} \geq n - r_1 + 1$  and so it must be  $\bar{j} = n - r_1 + 1$ . This implies that the inequality in (6) is an equality, which means that:

$$\Delta M_Z^{(q_1, j)} = \Delta M_Y^{(q_1, j)} - 1$$

for any  $j \geq \bar{j} = n - r_1 + 1$ , with  $j \leq n$ , and by (5)  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)} = 0$  for any  $(i, j) \geq (q_1 + 1, n - r_1 + 1)$ .

**Step 6.** If  $r \geq 2$ ,  $q_{r-1} = q_r$  and for any  $k \in \{1, \dots, r\}$  and  $i \geq q_k$   $\Delta M_X^{(i, n-k+1)} = 0$ , then  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)} = 0$  for any  $(i, j) \geq (q_k + 1, n - k + 1)$  and  $k \in \{1, \dots, r\}$  and  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)} - 1$  for any  $(i, j) \in \{(q_1, n), (q_2, n-1), \dots, (q_r, n-r+1)\}$ .

We proceed iterating the procedure given in Step 5. So let us suppose that for some  $h \in \{2, \dots, t+1\}$  the equalities in the claim hold for any  $i < q_{r_{h-1}+1}$  and for any  $j \geq n - r_{h-1} + 1$ . We will show that they hold also for  $q_{r_{h-1}+1} \leq i < q_{r_h} + 1$  and for any  $j \geq n - r_h + 1$ .

To this end, we repeat what we did in Step 5. So, as done before, we see that  $\sum_{j \leq n} \Delta M_Z^{(q_{r_{h-1}+1}-1, j)} - \sum_{j \leq n} \Delta M_Z^{(q_{r_{h-1}+1}, j)}$  is the number of  $(0, 1)$ -lines containing precisely  $q_{r_{h-1}+1}$  points of  $Z$ , while  $\sum_{j \leq n} \Delta M_Y^{(q_{r_{h-1}+1}-1, j)} - \sum_{j \leq n} \Delta M_Y^{(q_{r_{h-1}+1}, j)}$  is the number of  $(0, 1)$ -lines containing precisely  $q_{r_{h-1}+1}$  points of  $Y$ . By hypothesis it must be:

1) if  $q_{r_{h-1}+1} - 1 > q_{r_{h-1}}$ :

$$\begin{aligned} \sum_{j \leq n} \Delta M_Z^{(q_{r_{h-1}+1}-1, j)} - \sum_{j \leq n} \Delta M_Z^{(q_{r_{h-1}+1}, j)} &= \\ &= \sum_{j \leq n} \Delta M_Y^{(q_{r_{h-1}+1}-1, j)} - \sum_{j \leq n} \Delta M_Y^{(q_{r_{h-1}+1}, j)} + r_h - r_{h-1}; \end{aligned}$$

2) if  $q_{r_{h-1}+1} - 1 = q_{r_{h-1}}$ :

$$\begin{aligned} \sum_{j \leq n} \Delta M_Z^{(q_{r_{h-1}+1}-1, j)} - \sum_{j \leq n} \Delta M_Z^{(q_{r_{h-1}+1}, j)} &= \\ &= \sum_{j \leq n} \Delta M_Y^{(q_{r_{h-1}+1}-1, j)} - \sum_{j \leq n} \Delta M_Y^{(q_{r_{h-1}+1}, j)} + r_h - r_{h-1} - (r_{h-1} - r_{h-2}). \end{aligned}$$

By what we proved in Step 1 and Step 2 and by inductive hypothesis we see that these equalities are both equivalent to the following:

$$(7) \quad \sum_{n-r_h+1}^{n-r_{h-1}} \Delta M_Z^{(q_{r_{h-1}+1}, j)} = \sum_{n-r_h+1}^{n-r_{h-1}} \Delta M_Y^{(q_{r_{h-1}+1}, j)} - r_h + r_{h-1}.$$

By this equality and by Step 3 we see that there exists  $n-r_h+1 \leq \bar{j} \leq n-r_{h-1}$  such that  $\Delta M_Z^{(q_{r_{h-1}+1}, j)} = \Delta M_Y^{(q_{r_{h-1}+1}, j)}$  for any  $j < \bar{j}$  and  $\Delta M_Z^{(q_{r_{h-1}+1}, \bar{j})} < \Delta M_Y^{(q_{r_{h-1}+1}, \bar{j})}$ . In particular,  $\Delta M_Z^{(q_{r_{h-1}+1}, \bar{j})} \leq 0$  and so by Theorem 2.3 we have:

$$\Delta M_Z^{(i, \bar{j})} \leq 0$$

for any  $i \geq q_{r_{h-1}+1}$ . By Step 4 we have:

$$\begin{aligned} \sum_{i=q_{r_{h-1}+1}}^{a+1} \Delta M_Z^{(i, \bar{j})} &= \Delta M_Y^{(q_{r_{h-1}+1}, \bar{j})} - 1 \\ \Rightarrow 0 &\geq \sum_{i=q_{r_{h-1}+1}+1}^{a+1} \Delta M_Z^{(i, \bar{j})} = \Delta M_Y^{(q_{r_{h-1}+1}, \bar{j})} - \Delta M_Z^{(q_{r_{h-1}+1}, \bar{j})} - 1 \geq 0. \end{aligned}$$

This means that  $\Delta M_Y^{(q_{r_{h-1}+1}, \bar{j})} - \Delta M_Z^{(q_{r_{h-1}+1}, \bar{j})} - 1 = 0$  and that  $\Delta M_Z^{(i, \bar{j})} = 0$  for any  $i \geq q_{r_{h-1}+1} + 1$ .

Now take any  $j > \bar{j}$ . By the fact that  $\Delta M_Z^{(i, \bar{j})} = 0$  for any  $i \geq q_{r_{h-1}+1} + 1$  and by Theorem 2.3 we can say that  $\Delta M_Z^{(i, j)} \leq 0$  for any  $i \geq q_{r_{h-1}+1} + 1$  and any  $j > \bar{j}$ . By Step 4 we get:

$$(8) \quad 0 \geq \sum_{i=q_{r_{h-1}+1}+1}^{a+1} \Delta M_Z^{(i, j)} = \Delta M_Y^{(q_{r_{h-1}+1}, j)} - \Delta M_Z^{(q_{r_{h-1}+1}, j)} - 1$$

so that:

$$\Delta M_Z^{(q_{r_{h-1}+1}, j)} \geq \Delta M_Y^{(q_{r_{h-1}+1}, j)} - 1$$

for any  $j > \bar{j}$ . So we can say that:

$$(9) \quad \sum_{j=n-r_h+1}^{n-r_{h-1}} \Delta M_Z^{(q_{r_{h-1}+1}, j)} \geq \sum_{j=n-r_h+1}^{n-r_{h-1}} \Delta M_Y^{(q_{r_{h-1}+1}, j)} - n + r_{h-1} + \bar{j} - 1.$$

This fact compared to (7) gives us that  $\bar{j} \leq n-r_h+1$ , but by hypothesis  $\bar{j} \geq n-r_h+1$  and so it must be  $\bar{j} = n-r_h+1$ . This implies that the inequality in (9) is an equality, which means that:

$$\Delta M_Z^{(q_{r_{h-1}+1}, j)} = \Delta M_Y^{(q_{r_{h-1}+1}, j)} - 1$$

for any  $j \geq \bar{j} = n-r_h+1$ , with  $j \leq n-r_{h-1}$ , and by (8) and by inductive hypothesis  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)} = 0$  for any  $i \geq q_{r_{h-1}+1}$  and any  $j \geq n-r_h+1$ .

In this way we have proved the conclusion holds for any  $(i, j)$ , with  $i < q_{r_h+1}$  and the proof works by iteration.

**Step 7.** *If either  $r = 1$  or  $r \geq 2$ ,  $q_{r-1} < q_r$  and for any  $k \in \{1, \dots, r-1\}$  and  $i \geq q_k$   $\Delta M_X^{(i, n-k+1)} = 0$ , then  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)} = 0$  for any  $(i, j) > (q_k, n-k+1)$  and  $k \in \{1, \dots, r\}$  and  $\Delta M_Z^{(i, j)} = \Delta M_Y^{(i, j)} - 1$  for any  $(i, j) \in \{(q_1, n), (q_2, n-1), \dots, (q_r, n-r+1)\}$ .*



Let us first suppose that  $r \geq 2$  and that  $q_{r-1} < q_r$ . In this case the procedure given in Step 6 can be repeated for any  $h \in \{2, \dots, t\}$ . This means that the equalities in the conclusion of Step 7 hold for any  $i < q_r$ , for any  $j \geq n - r + 2$  and also for any  $j \leq n - r$  by Step 1, i.e. for any  $j \neq n - r + 1$ .

If  $r = 1$ , then by Step 1 and Remark 2.4 we see that the conclusion holds for any  $(i, j)$  with  $j \neq n - r + 1$  and for  $(i, n)$  with  $i < q_1$ . So in both cases we will show that  $\Delta M_Z^{(q_r, n-r+1)} = \Delta M_Y^{(q_r, n-r+1)} - 1$  and that  $\Delta M_Z^{(i, n-r+1)} = \Delta M_Y^{(i, n-r+1)} = 0$  for  $i > q_r$ .

As done before, we see that  $\sum_{j \leq n} \Delta M_Z^{(q_r-1, j)} - \sum_{j \leq n} \Delta M_Z^{(q_r, j)}$  is the number of  $(0, 1)$ -lines containing precisely  $q_r$  points of  $Z$ , while  $\sum_{j \leq n} \Delta M_Y^{(q_r-1, j)} - \sum_{j \leq n} \Delta M_Y^{(q_r, j)}$  is the number of  $(0, 1)$ -lines containing precisely  $q_r$  points of  $Y$ . By hypothesis it must be:

$$\sum_{j \leq n} \Delta M_Z^{(q_r-1, j)} - \sum_{j \leq n} \Delta M_Z^{(q_r, j)} = \sum_{j \leq n} \Delta M_Y^{(q_r-1, j)} - \sum_{j \leq n} \Delta M_Y^{(q_r, j)} + 1.$$

Since the equalities in the claim hold for any  $i < q_r$  and for any  $j \neq n - r + 1$ , we see that this equality is equivalent to the following:

$$(10) \quad \Delta M_Z^{(q_r, n-r+1)} = \Delta M_Y^{(q_r, n-r+1)} - 1.$$

Since the  $(0, 1)$ -lines containing exactly  $q_r + 1$  points of  $Z$  are one less than those containing exactly  $q_r + 1$  points of  $Y$ , we see that:

$$\sum_{j \leq n} \Delta M_Z^{(q_r, j)} - \sum_{j \leq n} \Delta M_Z^{(q_r+1, j)} = \sum_{j \leq n} \Delta M_Y^{(q_r, j)} - \sum_{j \leq n} \Delta M_Y^{(q_r+1, j)} - 1,$$

which, by our hypotheses, implies:

$$\Delta M_Z^{(q_r+1, n-r+1)} = \Delta M_Y^{(q_r+1, n-r+1)}.$$

By iterating the procedure, taken any  $i \geq q_r + 2$ , the  $(0, 1)$ -lines containing exactly  $i$  points of  $Z$  are also those containing exactly  $i$  points of  $Y$ , so that:

$$\sum_{j \leq n} \Delta M_Z^{(i-1, j)} - \sum_{j \leq n} \Delta M_Z^{(i, j)} = \sum_{j \leq n} \Delta M_Y^{(i-1, j)} - \sum_{j \leq n} \Delta M_Y^{(i, j)},$$

which, by our hypotheses, implies:

$$\Delta M_Z^{(i, n-r+1)} = \Delta M_Y^{(i, n-r+1)}.$$

□

In the same way, with the above notation, we can prove the following theorem:

**Theorem 3.2.** *Let  $C$  be a  $(0, 1)$ -line disjoint from  $X$ . Let  $R_{a+1}, \dots, R_m$ ,  $m \geq a$ , be arbitrary  $(1, 0)$ -lines and  $j_1, \dots, j_r \in \{0, \dots, a\}$ . Let  $\mathcal{P} = \{C \cap R_j \mid j \in \{0, \dots, m\}, j \neq j_1, \dots, j_r\}$  and let  $Z = X \cup \mathcal{P}$ . Suppose also that on the  $(1, 0)$ -line  $R_{j_k}$  there are  $p_k$  points of  $X$  for  $k = 1, \dots, r$  and that  $p_1 \leq p_2 \leq \dots \leq p_r$ . Then, given  $T = \{(m, p_1), (m-1, p_2), \dots, (m-r+1, p_r)\}$ , we have:*

$$\Delta M_Z^{(i, j)} = \begin{cases} 1 & \text{if } i \leq m, j = 0 \\ 0 & \text{if } i \geq m+1, j = 0 \\ \Delta M_X^{(i, j-1)} & \text{if } j \geq 1 \text{ and } (i, j) \notin T \\ \Delta M_X^{(i, j-1)} - 1 & \text{if } j \geq 1 \text{ and } (i, j) \in T \end{cases}$$

if one of the following conditions holds:

- (1)  $r = 1$ ;
- (2)  $r \geq 2$ ,  $p_{r-1} < p_r$  and for any  $k \in \{1, \dots, r-1\}$  and  $j \geq p_k$   $\Delta M_X^{(m-k+1, j)} = 0$ ;
- (3)  $r \geq 2$ ,  $p_{r-1} = p_r$  and for any  $k \in \{1, \dots, r\}$  and  $j \geq p_k$   $\Delta M_X^{(m-k+1, j)} = 0$ .

*Proof.* The proof works as in Theorem 3.1.  $\square$

Under the notation of Theorem 3.1 we prove the following:

**Theorem 3.3.** *If one the following conditions holds:*

- (1)  $q_{r-1} < q_r$  and  $n \geq b + r - 1$ ,
- (2)  $q_{r-1} = q_r$  and  $n \geq b + r$ ,

*then:*

$$\Delta M_Z^{(i, j)} = \begin{cases} 1 & \text{if } i = 0, j \leq n \\ 0 & \text{if } i = 0, j \geq n + 1 \\ \Delta M_X^{(i-1, j)} & \text{if } i \geq 1 \text{ and } (i, j) \notin T \\ \Delta M_X^{(i-1, j)} - 1 & \text{if } i \geq 1 \text{ and } (i, j) \in T. \end{cases}$$

*Proof.*

- (1) By Remark 2.4 our hypothesis imply that  $\Delta M_X^{(i, j)} = 0$  for any  $i \geq 0$  and for any  $j \geq n - r + 2$ , so that the hypothesis of Theorem 3.1 holds;
- (2) in this case by hypothesis we have that  $\Delta M_X^{(i, j)} = 0$  for any  $i \geq 0$  and for any  $j \geq n - r + 1$ , so that the hypothesis of Theorem 3.1 holds.

$\square$

In the same way under the notation of Theorem 3.2 we prove the following result:

**Theorem 3.4.** *If one the following conditions holds:*

- (1)  $p_{r-1} < p_r$  and  $m \geq a + r - 1$ ,
- (2)  $p_{r-1} = p_r$  and  $m \geq a + r$ ,

*then:*

$$\Delta M_Z^{(i, j)} = \begin{cases} 1 & \text{if } i \leq m, j = 0 \\ 0 & \text{if } i \geq m + 1, j = 0 \\ \Delta M_X^{(i, j-1)} & \text{if } j \geq 1 \text{ and } (i, j) \notin T \\ \Delta M_X^{(i, j-1)} - 1 & \text{if } j \geq 1 \text{ and } (i, j) \in T. \end{cases}$$

*Proof.* The works as in Theorem 3.3.  $\square$

*Example 3.5.* In these examples we will show that if the hypothesis of Theorem 3.1 does not hold, then the conclusion is not necessarily true. As a notation, we represent the  $(1, 0)$ -lines as horizontal lines and the  $(0, 1)$ -lines as vertical lines.

- (1) Let us consider a scheme  $X$  union of three generic points and its first difference  $\Delta M_X$ .

	0	1	2	3	4
0	1	1	1	0	...
1	1	0	-1	0	...
2	1	-1	0	0	...
3	0	0	0	0	...
4	...	...	...	...	...

$$\Delta M_X$$

Let  $R$  be a  $(1, 0)$ -line disjoint from  $X$  and let  $Z$  be the following scheme.

	0	1	2	3	4
0	1	1	1	0	...
1	1	1	-1	0	...
2	1	-1	0	0	...
3	1	-1	0	0	...
4	0	0	0	0	...
5	...	...	...	...	...

 $\Delta M_Z$ 

In this case, under the notation of Theorem 3.1 we have  $r = 2$ ,  $n = 2$ ,  $q_1 = q_2 = 1$  and  $\Delta M_X(q_1, n) = \Delta M_X(1, 2) \neq 0$ . In this case, we see that  $\Delta M_Z^{(1,2)} = -1 \neq \Delta M_X^{(0,2)} - 1 = 0$ .

- (2) Let us consider a scheme  $X$  of degree 4 with 2 points on a  $(1, 0)$ -line  $R_0$  and other 2 points on a  $(0, 1)$ -line  $C_0$  and its first difference  $\Delta M_X$ .

	0	1	2	3	4
0	1	1	1	0	...
1	1	0	0	0	...
2	1	0	-1	0	...
3	0	0	0	0	...
4	...	...	...	...	...

$\Delta M_X$

	0	1	2	3	4
0	1	1	1	0	...
1	1	1	0	0	...
2	1	0	-1	0	...
3	1	-1	0	0	...
4	0	0	0	0	...
5	...	...	...	...	...

$$\Delta M_Z$$

$$\Delta M_X^{(i,j)} = \begin{cases} 1 & \text{if } i \leq q_j - 1 \text{ and } 0 \leq j \leq b \\ 0 & \text{otherwise} \end{cases}$$

or equivalently:

$$\Delta M_X^{(i,j)} = \begin{cases} 1 & \text{if } j \leq p_i - 1 \text{ and } 0 \leq i \leq a \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We show that:

$$\Delta M_X^{(i,j)} = \begin{cases} 1 & \text{if } i \leq q_j - 1 \text{ and } 0 \leq j \leq b \\ 0 & \text{otherwise} \end{cases}$$

The proof that also:

$$\Delta M_X^{(i,j)} = \begin{cases} 1 & \text{if } j \leq p_i - 1 \text{ and } 0 \leq i \leq a \\ 0 & \text{otherwise} \end{cases}$$

is similar.

It is well known (see, for example, [1], [3] and [5]) that  $X$  can be described after a suitable permutation of lines in such a way that the following conditions holds:

- (1) for every  $i \in \{0, \dots, a\}$  there exists  $j(i) \in \{0, \dots, b\}$  such that  $R_i \cap C_j \in X$  for  $j \in \{0, \dots, j(i)\}$  and  $R_i \cap C_j \notin X$  for  $j \in \{j(i) + 1, \dots, b\}$ ;
- (2)  $j(0) \geq j(1) \geq \dots \geq j(a)$ .

Moreover, if any scheme  $X$  satisfies these conditions, then  $X$  is an ACM scheme. Using this fact we can easily compute  $\Delta M_X$  by induction on  $a$ . If  $a = 0$ , then the equality follows by the fact that  $h^0(\mathcal{I}_X(0, b)) = 0$  and  $h^0(\mathcal{I}_X(0, b + 1)) = 1$ .

If the equality holds for  $a - 1$ , then we apply Theorem 2.6 and we get the equality.  $\square$

**Theorem 4.2.** *Let  $X$  be an ACM scheme and let  $R$  be a  $(1, 0)$ -line disjoint from  $X$ . Let  $C_{b+1}, \dots, C_n$ ,  $n \geq b$ , be arbitrary  $(0, 1)$ -lines and  $i_1, \dots, i_r \in \{0, \dots, b\}$ . Let  $\mathcal{P} = \{R \cap C_i \mid i \in \{0, \dots, n\}, i \neq i_1, \dots, i_r\}$  and let  $Z = X \cup \mathcal{P}$ . Suppose also that on the  $(0, 1)$ -line  $C_{i_k}$  there are  $q_k$  points of  $X$  for  $k = 1, \dots, r$  and that  $q_1 \leq q_2 \leq \dots \leq q_r$ . Then, given  $T = \{(q_1, n), (q_2, n - 1), \dots, (q_r, n - r + 1)\}$ , we have:*

$$\Delta M_Z^{(i,j)} = \begin{cases} 1 & \text{if } i = 0, j \leq n \\ 0 & \text{if } i = 0, j \geq n + 1 \\ \Delta M_X^{(i-1,j)} & \text{if } i \geq 1 \text{ and } (i, j) \notin T \\ \Delta M_X^{(i-1,j)} - 1 & \text{if } i \geq 1 \text{ and } (i, j) \in T. \end{cases}$$

*Proof.* The conclusion follows by Theorem 3.1, by Proposition 4.1 and by the fact that  $\#(C_{b-k+1} \cap X) \leq q_k$ .  $\square$

In the same way:

**Theorem 4.3.** *Let  $X$  be an ACM scheme and let  $C$  be a  $(0, 1)$ -line disjoint from  $X$ . Let  $R_{a+1}, \dots, R_m$ ,  $m \geq a$ , be arbitrary  $(1, 0)$ -lines and  $j_1, \dots, j_r \in \{0, \dots, a\}$ . Let  $\mathcal{P} = \{C \cap R_j \mid j \in \{0, \dots, m\}, j \neq j_1, \dots, j_r\}$  and let  $Z = X \cup \mathcal{P}$ . Suppose also that on the  $(1, 0)$ -line  $R_{j_k}$  there are  $p_k$  points of  $X$  for  $k = 1, \dots, r$  and that  $p_1 \leq p_2 \leq \dots \leq p_r$ . Then, given  $T = \{(m, p_1), (m - 1, p_2), \dots, (m - r + 1, p_r)\}$ , we*

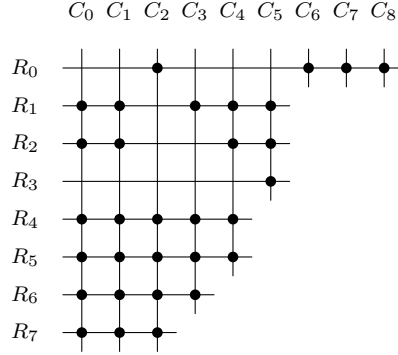
have:

$$\Delta M_Z^{(i,j)} = \begin{cases} 1 & \text{if } i \leq m, j = 0 \\ 0 & \text{if } i \geq m+1, j = 0 \\ \Delta M_X^{(i,j-1)} & \text{if } j \geq 1 \text{ and } (i,j) \notin T \\ \Delta M_X^{(i,j-1)} - 1 & \text{if } j \geq 1 \text{ and } (i,j) \in T. \end{cases}$$

*Proof.* The works as in Theorem 3.3.  $\square$

## 5. EXAMPLE

Now we show how it is possible to apply Theorem 3.1, Theorem 3.3 and Theorem 4.2 to compute the first difference of the Hilbert matrix of a scheme  $X$  whose points can be distributed on a grid of  $(1,0)$  and  $(0,1)$ -lines in the following way:

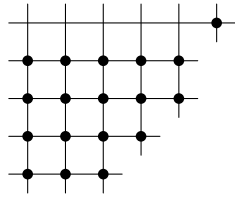


The scheme  $X$

We compute  $\Delta M_X$  by adding the points of the  $(1,0)$ -lines. The points on  $R_4$ ,  $R_5$ ,  $R_6$  and  $R_7$  are an aCM scheme, so that, by using Proposition 4.1, we get its first difference:

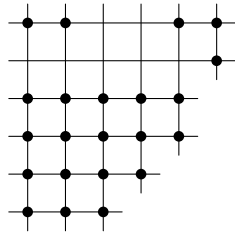
	0	1	2	3	4	5	6
0	1	1	1	1	1	0	...
1	1	1	1	1	1	0	...
2	1	1	1	1	0	0	...
3	1	1	1	0	0	0	...
4	0	0	0	0	0	0	...
5	...	...	...	...	...	...	...

Now we add the point on the line  $R_3$  and by Theorem 4.2 we compute its first difference:



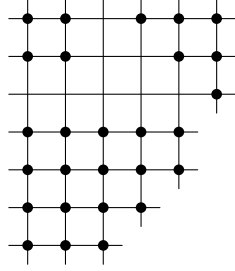
	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	0	...
1	1	1	1	1	1	0	0	...
2	1	1	1	1	1	-1	0	...
3	1	1	1	1	-1	0	0	...
4	1	0	0	-1	0	0	0	...
5	0	0	0	0	0	0	0	...
6	...	...	...	...	...	...	...	...

In the same way we add the points on  $R_2$  and by Theorem 3.1 we compute the first difference:



	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	0	...
1	1	1	1	1	1	1	0	...
2	1	1	1	1	1	0	0	...
3	1	1	1	1	1	-2	0	...
4	1	1	1	1	-2	0	0	...
5	1	0	0	-1	0	0	0	...
6	0	0	0	0	0	0	0	...
7	...	...	...	...	...	...	...	...

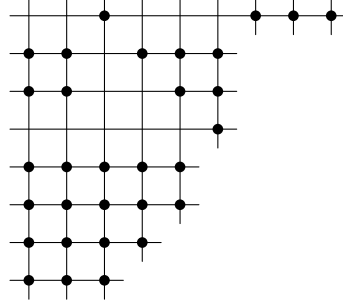
Now we add the points on  $R_1$  and again by Theorem 3.1 we compute its first difference:



	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	0	...
1	1	1	1	1	1	1	0	...
2	1	1	1	1	1	1	0	...
3	1	1	1	1	1	0	0	...
4	1	1	1	1	1	-3	0	...
5	1	1	1	1	-2	0	0	...
6	1	0	0	-1	0	0	0	...
7	0	0	0	0	0	0	0	...
8	...	...	...	...	...	...	...	...

Finally, by applying again Theorem 3.1 we get the first difference  $\Delta M_X$  of  $X$ .





	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	0	...
1	1	1	1	1	1	1	0	0	0	0	...
2	1	1	1	1	1	1	0	0	0	0	...
3	1	1	1	1	1	1	0	0	-1	0	...
4	1	1	1	1	1	0	-1	-1	0	0	...
5	1	1	1	1	1	-3	0	0	0	0	...
6	1	1	1	1	-3	-1	0	0	0	0	...
7	1	0	0	-1	0	0	0	0	0	0	...
8	0	0	0	0	0	0	0	0	0	0	...
9	...	...	...	...	...	...	...	...	...	...	...

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